On the Maximum Buffer Size Achieved in a Class of Constructions of Optical Priority Queues

Jay Cheng, Shin-Shiang Huang, Hsin-Hung Chou, and Ming-Che Tang

***Abstract*—The design of optical buffers is an important issue in all-optical packet switching. One of the most general types of buffering schemes is priority queues, which includes first-in first-out (FIFO) queues and last-in first-out (LIFO) queues as special cases (where the packet arrival times are used for the assignment of packet priorities). Recently, it was shown in our previous work that an optical priority queue with buffer size $2^{O(\sqrt{\alpha M})}$ can be implemented by using an optical $(M+2) \times (M+2)$ (bufferless) crossbar switch and M fiber delay lines under a simple priority-based routing policy, where α is a constant that depends on the parameters used in the constructions. This achieved buffer size $2^{O(\sqrt{\alpha M})}$ (which is exponential in $\sqrt{M}$) is the best result currently known in the literature and significantly improves on all previous results (all of which are only polynomial in M). In this paper, we focus on our previous constructions of optical priority queues. The first contribution of this paper is to derive a closed-form expression for the maximum buffer size that can be achieved in our previous constructions. Such an expression is of sufficient theoretical interest itself and can be used to *directly* compute the maximum buffer size (in contrast, the maximum buffer size has to be computed *recursively* in our previous work). The second contribution of this paper is to use the closed-form expression to show that in the regime that $s \geq 2$, $k \geq 2s+1$, and $m \geq 2$, where s, k, and m are parameters used in the constructions, the maximum buffer size U_k is given by $U_k = 2^{O(\sqrt{M \log_2 (2s+2) \log_2 m/((2s+1)m)})}$ under a mild constraint that is applicable in practical scenarios. This result can be regarded as a complement to the approximate result $U_k \approx 2^{O(\sqrt{M \log_2 (2s+2) \log_2 m/((2s+1)m)})}$ in our previous work.**

***Index Terms*—FIFO multiplexers, optical buffers, optical queues, optical switches, priority queues.**

I. INTRODUCTION

THE design and implementation of optical buffers for contention resolution among packets competing for the same resources has been well recognized as an important and challenging issue in all-optical packet switching. One of the feasible approaches for the implementation of optical buffers has the following features: (i) Using optical fiber delay lines as storage media to store optical packets. (ii) Using optical (bufferless) crossbar switches to route optical packets through the fiber delay lines.

Manuscript received August 2, 2022 revised January 19, 2023; approved for publication by Anna Maria Vegni, Division 1 Editor, March 19, 2023.

This work was supported in part by the Ministry of Science and Technology, Taiwan, under Contract MOST-111-2221-E-007-041-MY2.

The authors are with the Department of Electrical Engineering, National Tsing Hua University, Hsinchu 30013, Taiwan, e-mail: jcheng@ee.nthu.edu.tw; whu910116hung1009@gmail.com; d9564808@oz.nthu.edu.tw; TommyT0105@gmail.com.

J. Cheng is the corresponding author.

Digital Object Identifier: 10.23919/JCN.2023.000011

We note that optical packets are constantly moving along the fibers into which they are routed, and they cannot be accessed until they reach the outputs of the fibers. As a result, such an approach by using optical fiber delay lines as storage media does not have random-access capability. Therefore, the delays of the optical fiber delay lines have to be properly chosen and the routing policy performed by the optical (bufferless) crossbar switches also has to be carefully designed. By so doing, packets can be routed to the right places at the right times, and exact emulations of the desired optical buffers can be achieved.

In the last three decades, there have been extensive studies on the constructions of optical buffers by using the switched-delay-lines (SDL) approach described above. Indeed, such an SDL approach has been successfully used to construct a variety of optical buffers in the literature. These works include: (i) The early feasibility studies in [1]–[4], (ii) output-buffered switches in [5]–[10], (iii) first-in first-out (FIFO) multiplexers in [5] and [10]–[20], (iv) FIFO queues in [20]–[25], (v) last-in first-out (LIFO) queues in [22], [23], and [26], (vi) priority queues in [27]–[36], (vii) time slot interchanges in [20] and [37], (viii) linear compressors/decompressors, non-overtaking delay lines, and flexible delay lines in [20] and [38]–[43], and (ix) FIFO/LIFO/absolute contractors in [44]. Furthermore, results on the fundamental complexity of SDL constructions of optical queues can be found in [45] and performance analysis for optical queues has been addressed in [46] and [47].

In this paper (as well as many of the works in the literature), we focus on the theoretical aspect of the constructions of optical buffers. We are aware of many important practical feasibility issues such as: (i) Router buffer sizing problem, (ii) fault-tolerant capability, and (iii) limitation on the number of times that an optical packet can recirculate through optical switches and fiber delay lines. For those who are interested in such practical feasibility issues, we refer to Sections V-A and V-C in [36] and the references therein for details. For review articles on SDL constructions of optical buffers as well as related implementation and feasibility issues, we refer to [48]–[53] and the references therein.

One of the most general types of buffering schemes is priority queues, which includes FIFO queues and LIFO queues as special cases. A priority queue can be described as follows: (i) Each packet is associated with a unique priority upon its arrival, (ii) the packet with the *highest* priority is sent out from the queue whenever there is a departure request and there are packets in the queue, and (iii) the packet with the *lowest* priority is dropped from the queue whenever there is a buffer overflow. We note that the assignment of packet priorities can

1229-2370/23/$10.00 © 2023 KICS

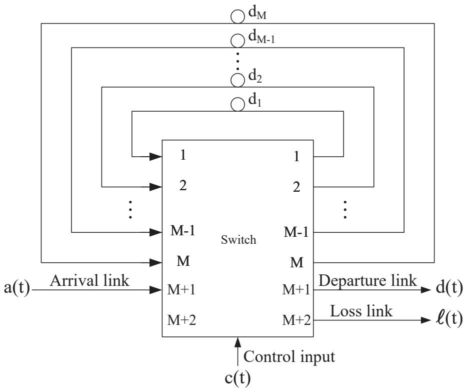

Fig. 1. A construction of an optical priority queue in [27] by using a feedback system consisting of an optical $(M+2)\times(M+2)$ (bufferless) crossbar switch and M fiber delay lines with delays $d_1, d_2, \cdots, d_M$.

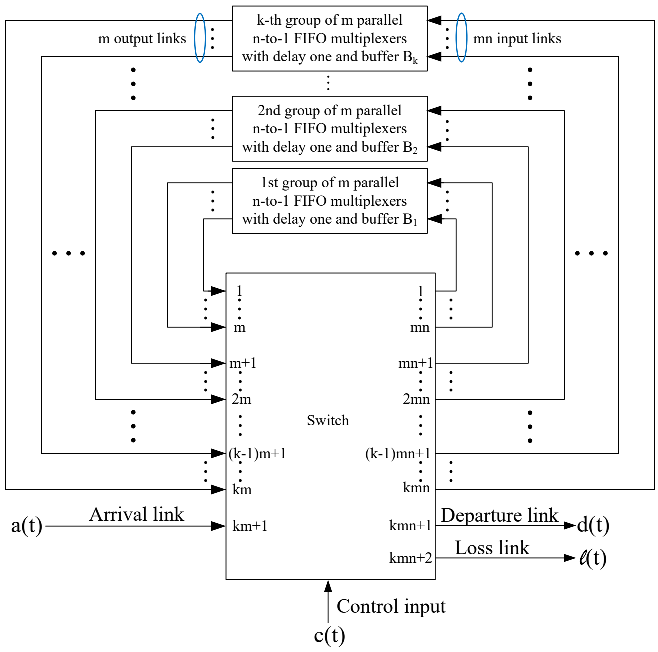

Fig. 2. A construction of an optical priority queue in [36] by using an optical $(kmn+2)\times(kmn+2)$ (bufferless) crossbar switch and k groups of m parallel optical n-to-1 FIFO multiplexers with delay one (nFM1's).

be arbitrary subject to the constraint that every packet in the queue has a distinct priority and the relative priority order between any two packets remains unchanged as long as they are in the queue. In the special case of a FIFO (resp., LIFO) queue, the packet arrival times are used for the assignment of packet priorities so that a packet with earlier arrival time has higher (resp., lower) priority than a packet with later arrival time.

The first construction of optical priority queues was given in [27] by Sarwate and Anantharam of UC Berkeley more than one and a half decades ago. It was shown in [27] that an optical priority queue with buffer size $O(M^2)$ can be constructed by using a feedback system consisting of an optical $(M+2)\times(M+2)$ (bufferless) crossbar switch and M fiber delay lines (see Fig. 1). Since the publication of the paper [27], there have been several works [28]–[36] on the constructions of optical priority queues that improve on the $O(M^2)$ buffer size. The buffer size achieved in [36] is $2^{O(\sqrt{\alpha M})}$, where α is a constant that depends on the parameters used in the constructions in [36]. This buffer size $2^{O(\sqrt{\alpha M})}$ (which is exponential in $\sqrt{M}$) dramatically outperforms all previous results in [27]–[35] (all of which are only polynomial in M) and is the best result currently available in the literature.

In this paper, we focus on the constructions of optical priority queues in our previous work [36]. The main contributions of this paper are as follows:

(i) We derive a closed-form expression for the maximum buffer size U_k (see (14) and (15) in Theorem 4 in Section III-A) that can be achieved in the constructions in [36]. Such a closed-form expression is not only of sufficient theoretical interest itself, but also can be used to *directly* compute the maximum buffer size. In contrast, the maximum buffer size has to be computed *recursively* in [36].

(ii) We use the closed-form expression to show that in the regime that $s \geq 2$, $k \geq 2s+1$, and $m \geq 2$, where s, k, and m are parameters used in the constructions, the maximum buffer size U_k is given by $U_k = 2^{O(\sqrt{M \log_2(2s+2)\log_2 m/((2s+1)m)})}$ (see (24) in Theorem 6 in Section IV) under a mild constraint that is applicable in practical scenarios. As there is no closed-form expression available for the maximum buffer size in [36], an approximate analysis was resorted to in [36] to obtain the approximate result $U_k \approx 2^{O(\sqrt{M \log_2(2s+2)\log_2 m/((2s+1)m)})}$ in this regime. Therefore, our result in this paper complements the results in [36].

We note the motivation for deriving a closed-form expression for the maximum buffer size U_k in this paper is to use such an expression to give a rigorous mathematical proof that the approximate result $U_k \approx 2^{O(\sqrt{M \log_2(2s+2)\log_2 m/((2s+1)m)})}$ in [36] is indeed an exact result in the regime that $s \geq 2$, $k \geq 2s+1$, and $m \geq 2$. It turns out that the closed-form expression obtained in this paper holds for *all* regimes of the parameters s, k, and m, i.e., for all $1 \leq s \leq k-1$ and $m \geq 2$.

This paper is organized as follows. We first give in Section II a review of the constructions of optical priority queues in [36]. In Section III, we derive a closed-form expression for the maximum buffer size and present our numerical results on the maximum buffer size. Then in Section IV, we show that in the regime that $s \geq 2$, $k \geq 2s+1$, and $m \geq 2$, the maximum buffer size U_k is given by $U_k = 2^{O(\sqrt{M \log_2(2s+2)\log_2 m/((2s+1)m)})}$ under a mild constraint that is applicable in practical scenarios. Finally, we give a brief conclusion in Section V.

II. THE CONSTRUCTIONS IN OUR PREVIOUS WORK

In this section, we review the constructions of optical priority queues in [36]. The constructions in [36] use a feedback system (see Fig. 2) consisting of an optical $(kmn+2)\times(kmn+2)$ (bufferless) crossbar switch and k groups of optical n-to-1 FIFO multiplexers with delay one (nFM1's). The i th group in Fig. 2 has m parallel optical nFM1's with the same buffer size B_i ($B_i \geq 1$) for $i = 1, 2, \cdots, k$.

In [36], every packet that has to be buffered in the queue is associated with a distinct *buffering tag*, and each group of nFM1's in Fig. 2 is associated with a unique set of buffering tags. A packet at the input links of the crossbar switch in Fig. 2 that has to be buffered in the queue (i.e., the packet is not routed to the departure link or the loss link) is routed to the group of nFM1's whose associated set of buffering tags contains the buffering tag of the packet. Furthermore, packets routed to a group of nFM1's are distributed to the m nFM1's in that group in a *round-robin* fashion so that load balancing among the nFM1's in that group can be achieved and hence the buffering capacity of the nFM1's can be fully utilized. By so doing, the highest-priority (resp., lowest-priority) packet is always available at the input links of the crossbar switch whenever it needs to be routed to the departure (resp., loss) link. This is why an optical priority queue can be successfully constructed in [36].

Indeed, with appropriate choices of the parameters used in the constructions, a class of constructions of optical priority queues was obtained in Theorem 7 in [36]. By using the SDL constructions in [12] to implement the nFM1's in Fig. 2, it was further shown in [36] that the construction in Fig. 2 can be implemented by using a feedback system consisting of an optical $(M+2)\times(M+2)$ (bufferless) crossbar switch and M fiber delay lines as in Fig. 1. The maximum buffer size U_k that can be achieved in the class of constructions in [36] and the corresponding value of M in Fig. 1 are recalled in the following theorem.

Theorem 1 *[36, Theorem 9] Suppose $1 \le s \le k-1$ and $m \ge 2$. Then an optical priority queue with buffer size U_k can be constructed by using a feedback system consisting of an optical $(M+2)\times(M+2)$ (bufferless) crossbar switch and M fiber delay lines as in Fig. 1, where*

$$U_k = \sum_{i=1}^{k}((m-1)B_i + 1), \tag{1}$$

and

$$M = m\sum_{i=1}^{k}((n-1)\lceil \log_n B_i \rceil + n + 1), \tag{2}$$

in which $n = \min\{2s+1, k\}+1$ and $B_1, B_2, \cdots, B_k$ are given as follows: If $s+1 \le k \le 2s$, then $B_1 = B_k = 1$ and $B_2, B_3, \cdots, B_{k-1}$ are recursively given by

$$B_i = B_{k-i+1} = \sum_{j=1}^{i-1}((m-1)B_j + 1), \text{ if } 2 \le i \le \lceil k/2 \rceil. \tag{3}$$

On the other hand, if $k \ge 2s+1$, then $B_1 = B_k = 1$ and $B_2, B_3, \cdots, B_{k-1}$ are recursively given by

$$\begin{aligned} B_i &= B_{k-i+1} \\ &= \begin{cases} \sum_{j=1}^{i-1}((m-1)B_j + 1), & \text{if } 2 \le i \le s+1, \\ \sum_{j=i-s}^{i-1}((m-1)B_j + 1), & \text{if } s+2 \le i \le \lceil k/2 \rceil. \end{cases} \end{aligned} \tag{4}$$

As mentioned in Section I that the maximum buffer size U_k is given by $2^{O(\sqrt{\alpha M})}$, where α is a constant that depends on the parameters s, k, and m used in the constructions. We recall this result in the following theorem.

Theorem 2 *[36, Theorem 11] Suppose that $1 \le s \le k-1$, $m \ge 2$, U_k is given by (1), and M is given by (2).*

(i) If $s = 1$, $k \ge 3$, and $m \ge 3$, then we have

$$U_k = 2^{O(\sqrt{2M \log_2 (m-1)/(3m)})}. \tag{5}$$

(ii) If $s \ge 2$, $s+1 \le k \le 2s$, and $m \ge 2$, then we have

$$U_k = 2^{O(\sqrt{M \log_2 (k+1) \log_2 m/(km)})}. \tag{6}$$

(iii) If $s \ge 2$, $k \ge 2s+1$, and $m \ge 2$, then we have the following approximate result:

$$U_k \approx 2^{O(\sqrt{M \log_2 (2s+2) \log_2 m/((2s+1)m)})}. \tag{7}$$

For the two simple regimes that "$s = 1$, $k \ge 3$, and $m \ge 3$" and "$s \ge 2$, $s+1 \le k \le 2s$, and $m \ge 2$" in Theorem 2(i)(ii), $B_1, B_2, \cdots, B_k$ can be obtained in closed forms (see Theorem 10(i)(ii) in [36]), and hence the maximum buffer size U_k can be obtained as in (5) and (6) in these two regimes.

However, for the *broader* regime that "$s \ge 2$, $k \ge 2s+1$, and $m \ge 2$" in Theorem 2(iii), there is no closed-form expression available in [36] for $B_1, B_2, \cdots, B_k$. As a result, an approximation analysis was used in [36] as described below. It was first shown in Theorem 10(iii) in [36] that

$$B_i = B_{k-i+1} = \sum_{j=1}^{s}\alpha_j \lambda_j^i - \frac{s}{s(m-1)-1}, \tag{8}$$

for $1 \le i \le \lceil k/2 \rceil$. In (8), $\lambda_1, \lambda_2, \cdots, \lambda_s$ are the roots of the characteristic polynomial $p(z) = z^s - \sum_{j=0}^{s-1}(m-1)z^j$ associated with the s th-order nonhomogeneous linear difference equation with constant coefficients given by $B_i = \sum_{j=i-s}^{i-1}((m-1)B_j + 1)$ for $s+1 \le i \le \lceil k/2 \rceil$, and $\alpha_1, \alpha_2, \cdots, \alpha_s$ can be obtained by solving the s equations $B_1 = 1$ and $B_i = m^{i-1} + \frac{m^{i-2}-1}{m-1}$, $i = 2, 3, \cdots, s$.

Let λ_+ be the positive root of $p(z)$ (it was shown in Lemma 12 in the full version of the paper [36] that $p(z)$ has only one positive root) and let α_+ be the coefficient corresponding to λ_+ in (8). By approximating $B_1, B_2, \cdots, B_k$ by $B_i = B_{k-i+1} \approx \alpha_+ \lambda_+^i$ for $1 \le i \le \lceil k/2 \rceil$, and substituting the approximate B_i into (1), we have the following approximate result on U_k

$$\begin{aligned} &U_k \\ &\approx \widetilde{U}_k \\ &= \begin{cases} 2(m-1)\alpha_+\lambda_+ \frac{\lambda_+^{\ell}-1}{\lambda_+-1} + 2\ell, & \text{if } k = 2\ell, \\ (m-1)\alpha_+\lambda_+ \frac{\lambda_+^{\ell+1}+\lambda_+^{\ell}-2}{\lambda_+-1} + 2\ell + 1, & \text{if } k = 2\ell+1. \end{cases} \end{aligned} \tag{9}$$

By further approximating λ_+ by m, the approximate result on U_k in (7) was obtained in [36].

In the rest of the paper, we will derive a closed-form expression for $B_1, B_2, \cdots, B_k$ (see (13) in Theorem 4 in Section III-A), and substitute it into (1) to obtain a closed-form expression for the maximum buffer size U_k (see (14) and (15) in Theorem 4 in Section III-A) for *all* regimes of the parameters s, k, and m, i.e., for all $1 \leq s \leq k-1$ and $m \geq 2$. For the regime that $s \geq 2$, $k \geq 2s+1$, and $m \geq 2$, we will use these closed-form expressions to show that U_k is given by $U_k = 2^{O(\sqrt{M \log_2(2s+2)\log_2 m/((2s+1)m)})}$ (see (24) in Theorem 6 in Section IV) under a mild constraint that is applicable in practical scenarios.

III. A Closed-Form Expression for the Maximum Buffer Size

In this section, we derive a closed-form expression for the maximum buffer size and present our numerical results on the maximum buffer size.

A. The Closed-Form Expression

We first give a lemma that is the key to the derivation of the closed-form expression for the maximum buffer size in this paper.

Lemma 3 *Suppose that s is a positive integer and a is a positive real number. Assume that $x_1 = 1$ and x_i, $i \geq 2$, are recursively given as follows:*

$$x_i = \begin{cases} \sum_{j=1}^{i-1}(ax_j+1), & \text{if } 2 \leq i \leq s+1, \\ \sum_{j=i-s}^{i-1}(ax_j+1), & \text{if } i \geq s+2. \end{cases} \tag{10}$$

For $i \geq 1$, let q_i be the unique nonnegative integer such that $q_i(s+1)+1 \leq i \leq (q_i+1)(s+1)$, i.e., $q_i = \lceil i/(s+1) \rceil - 1$.

(i) If $i \geq 3$, then x_i can be recursively given as follows:

$$x_i = \begin{cases} (a+1)x_{i-1}+1, & \text{if } 3 \leq i \leq s+1, \\ (a+1)x_{i-1} - ax_{i-s-1}, & \text{if } i \geq s+2. \end{cases} \tag{11}$$

(ii) If $i \geq 2$, then x_i can be expressed in closed form as follows:

$$\begin{aligned} x_i = &\sum_{j=0}^{q_i}(-1)^j(1/j!) \\ &\times[j(i-j(s+1))_{j-1}a + (i-j(s+1))_j(a^2+a+1)] \\ &\times a^{j-1}(a+1)^{i-j(s+1)-2} - 1/a, \end{aligned} \tag{12}$$

where $(y)_j$ is the Pochhammer symbol given by $(y)_{-1} = (y)_0 = 1$ and $(y)_j = y(y+1)(y+2)\cdots(y+j-1)$ for every positive integer j.

(iii) The sequence $\{x_i\}_{i=1}^{\infty}$ is strictly increasing.

Proof. See Appendix A. ■

In the following theorem, we use Lemma 3 to derive closed-form expressions for $B_1, B_2, \cdots, B_k$ that are given by (3) and (4) and for the maximum buffer size U_k that is given by (1).

Theorem 4 *Suppose that $1 \leq s \leq k-1$ and $m \geq 2$. Let $x_1 = 1$ and let x_i be given by the closed-form expression in (12) (with $a = m-1$ in (12)) for $2 \leq i \leq \lceil k/2 \rceil$.*

(i) We have

$$B_i = B_{k-i+1} = x_i \text{ for } 1 \leq i \leq \lceil k/2 \rceil. \tag{13}$$

Therefore, we have $B_1 < B_2 < \cdots < B_{\lceil k/2 \rceil}$.

(ii) Suppose that k is even, say $k = 2\ell$ for some $\ell \geq 1$. Let q_ℓ be given as in Lemma 3. Then we have

$$U_k = \begin{cases} 2m\sum_{r=0}^{q_\ell} x_{\ell-r(s+1)} + 2q_\ell, \\ \quad \text{if } \ell = q_\ell(s+1)+1, \\ 2m\sum_{r=0}^{q_\ell} x_{\ell-r(s+1)} + 2q_\ell + 2, \\ \quad \text{if } q_\ell(s+1)+2 \leq \ell \leq (q_\ell+1)(s+1). \end{cases} \tag{14}$$

(iii) Suppose that k is odd, say $k = 2\ell+1$ for some $\ell \geq 1$ (note that $k \geq s+1 \geq 2$). Let q_ℓ be given as in Lemma 3. Then we have

$$\begin{aligned} &U_k \\ &= \begin{cases} 2m\sum_{r=0}^{q_\ell} x_{\ell-r(s+1)} + (m-1)x_{\ell+1} + 2q_\ell + 1, \\ \quad \text{if } \ell = q_\ell(s+1)+1, \\ 2m\sum_{r=0}^{q_\ell} x_{\ell-r(s+1)} + (m-1)x_{\ell+1} + 2q_\ell + 3, \\ \quad \text{if } q_\ell(s+1)+2 \leq \ell \leq (q_\ell+1)(s+1). \end{cases} \end{aligned} \tag{15}$$

Proof. (i) It is clear that (13) follows from (3), (4), (10), and Lemma 3(ii) (with $a = m-1 > 0$). From (13) and Lemma 3(iii), we obtain $B_1 < B_2 < \cdots < B_{\lceil k/2 \rceil}$.

(ii) From (1) and (13), we have

$$\begin{aligned} U_k &= \sum_{i=1}^{k}((m-1)B_i+1) = 2\sum_{j=1}^{\ell}((m-1)x_j+1) \\ &= 2\left[\sum_{j=1}^{\ell-q_\ell(s+1)}((m-1)x_j+1)\right. \\ &\quad \left.+\sum_{r=0}^{q_\ell-1}\sum_{j=\ell-(r+1)(s+1)+1}^{\ell-r(s+1)}((m-1)x_j+1)\right]. \end{aligned} \tag{16}$$

If $\ell = q_\ell(s+1)+1$, then we have

$$\begin{aligned} \sum_{j=1}^{\ell-q_\ell(s+1)}((m-1)x_j+1) &= (m-1)x_1+1 = m \\ &= mx_1 = mx_{\ell-q_\ell(s+1)}. \end{aligned} \tag{17}$$

On the other hand, if $q_\ell(s+1)+2 \leq \ell \leq (q_\ell+1)(s+1)$, then we have from (10) (with $a = m-1$) and $2 \leq \ell - q_\ell(s+1) \leq s+1$ that

$$\begin{aligned} &\sum_{j=1}^{\ell-q_\ell(s+1)}((m-1)x_j+1) \\ &= x_{\ell-q_\ell(s+1)} + ((m-1)x_{\ell-q_\ell(s+1)}+1) \\ &= mx_{\ell-q_\ell(s+1)}+1. \end{aligned} \tag{18}$$

For $0 \leq r \leq q_\ell - 1$, we have from (10) (with $a = m-1$) and $\ell - r(s+1) \geq q_\ell(s+1)+1-(q_\ell-1)(s+1) = s+2$ that

$$\begin{aligned} &\sum_{j=\ell-(r+1)(s+1)+1}^{\ell-r(s+1)}((m-1)x_j+1) \\ &= x_{\ell-r(s+1)} + ((m-1)x_{\ell-r(s+1)}+1) \\ &= mx_{\ell-r(s+1)}+1. \end{aligned} \tag{19}$$

By substituting (17)–(19) into (16), we obtain (14).

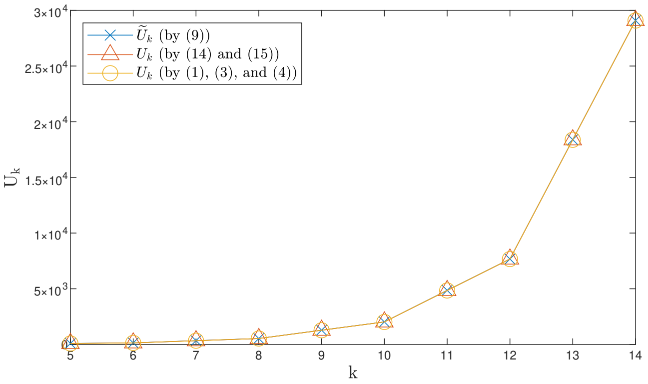

Fig. 3. The approximate maximum buffer size $\widetilde{U}_k$ given by (9), the maximum buffer size U_k computed directly by (14) and (15), and the maximum buffer size U_k computed recursively by (1), (3), and (4) for the case that $s = 2$, $m = 4$, and $5 \le k \le 14$.

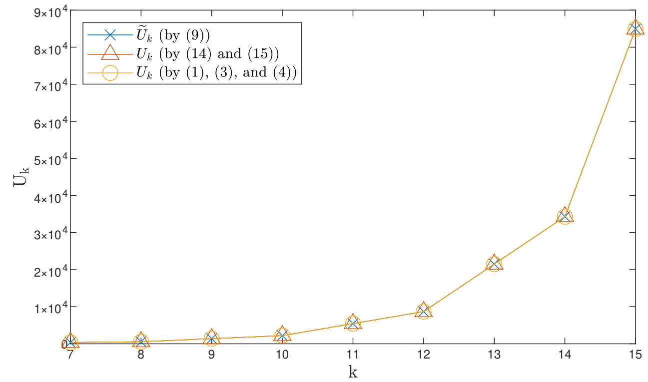

Fig. 4. The approximate maximum buffer size $\widetilde{U}_k$ given by (9), the maximum buffer size U_k computed directly by (14) and (15), and the maximum buffer size U_k computed recursively by (1), (3), and (4) for the case that $s = 3$, $m = 4$, and $7 \le k \le 15$.

(iii) From (1) and (13), we have

$$\begin{aligned} U_k &= \sum_{i=1}^{k}((m-1)B_i + 1) \\ &= 2\sum_{j=1}^{\ell}((m-1)x_j + 1) + (m-1)x_{\ell+1} + 1. \quad (20) \end{aligned}$$

As it is clear from (16) that $2\sum_{j=1}^{\ell}((m-1)x_j + 1)$ is given by the right-hand side of (14), we immediately see that (15) follows from (20) and (14). ∎

B. Numerical Results

We have performed numerical analysis for extensive ranges of the parameters s, k, and m, and our numerical results show that the maximum buffer size U_k computed directly by using the closed-form expressions in (14) and (15) is the same as that computed recursively by using (1), (3), and (4). This can be easily seen from Fig. 3 for the case that $s = 2$, $m = 4$, and $5 \le k \le 14$, and from Fig. 4 for the case that $s = 3$, $m = 4$, and $7 \le k \le 15$. Furthermore, these numerical results serve as a verification of the correctness of the closed-form expressions for U_k in (14) and (15).

TABLE I

THE APPROXIMATE MAXIMUM BUFFER SIZE $\widetilde{U}_k$ AND THE MAXIMUM BUFFER SIZE U_k FOR THE CASE THAT $s = 2$, $m = 4$, AND $5 \le k \le 19$.

$s = 2, m = 4$					
k	5	6	7	8	9
$\widetilde{U}_k$	91	140	341	533	1285
U_k	86	138	334	530	1275
$\widetilde{U}_k - U_k$	5	2	7	3	10
k	10	11	12	13	14
$\widetilde{U}_k$	2023	4856	7671	18390	29088
U_k	2020	4844	7668	18376	29084
$\widetilde{U}_k - U_k$	3	12	3	14	4
k	15	16	17	18	19
$\widetilde{U}_k$	69698	110282	264213	418114	1001672
U_k	69681	110278	264194	418110	1001650
$\widetilde{U}_k - U_k$	17	4	19	4	22

TABLE II

THE APPROXIMATE MAXIMUM BUFFER SIZE $\widetilde{U}_k$ AND THE MAXIMUM BUFFER SIZE U_k FOR THE CASE THAT $s = 3$, $m = 4$, AND $7 \le k \le 21$.

$s = 3, m = 4$					
k	7	8	9	10	11
$\widetilde{U}_k$	353	555	1382	2196	5446
U_k	346	554	1374	2194	5435
$\widetilde{U}_k - U_k$	7	1	8	2	11
k	12	13	14	15	16
$\widetilde{U}_k$	8678	21497	34294	84915	135511
U_k	8676	21484	34292	84900	135508
$\widetilde{U}_k - U_k$	2	13	2	15	3
k	17	18	19	20	21
$\widetilde{U}_k$	335498	535455	1325637	2115785	5238040
U_k	335480	535452	1325617	2115782	5238018
$\widetilde{U}_k - U_k$	18	3	20	3	22

Our numerical results also show that the approximate maximum buffer size $\widetilde{U}_k$ given by (9) is very close to the maximum buffer size U_k. In fact, $\widetilde{U}_k$ is slightly larger than U_k. This is due to the fact that when we approximate B_i given in (8) by $\alpha_+\lambda_+^i$, the terms in (8) that are omitted contribute negatively to the maximum buffer size. As a result, $\widetilde{U}_k$ is slightly larger than U_k. Since it is not easy to tell the difference between $\widetilde{U}_k$ and U_k visually from Figs. 3 and 4, we show the values of $\widetilde{U}_k$ and U_k in Table I and Table II. It can be easily seen from Table I and Table II that the difference between $\widetilde{U}_k$ and U_k is at most equal to 22 in these cases. These numerical results tell us that the approximate result in [36] gives a slight overestimate of the maximum buffer size U_k. Fortunately, the difference between $\widetilde{U}_k$ and U_k is insignificant in most scenarios when compared with the values of U_k, especially when s, k, and m are large.

The observation from the numerical results that the approximate maximum buffer size $\widetilde{U}_k$ is very close to the maximum buffer size U_k is a good indication that the approximate result in (7) is indeed an exact result as given in (24). In Section IV, we will use the closed-form expressions in (14) and (15) to prove that the exact result in (24) holds.

IV. The Maximum Buffer Size in Terms of the Switch Size in Fig. 1

To express the maximum buffer size U_k (which is given by (1)) in terms of the switch size M (which is given by (2)) in the regime that $s \geq 2$, $k \geq 2s+1$, and $m \geq 2$, we need the following lemma on an upper bound and a lower bound for the x_i's in Lemma 3 under the constraint that $(a+1)^{s+1} \geq q_i(s+1)a+1$.

Lemma 5 *Suppose that s, a, x_i, $i \geq 1$, and q_i, $i \geq 1$, are as given in Lemma 3. If $(a+1)^{s+1} \geq q_i(s+1)a+1$ for some $i \geq 1$, then we have the following upper bound and lower bound for x_i:*

$$x_i \leq (a+1)^i/a, \tag{21}$$

and

$$x_i \geq \begin{cases} (a+1)^{i-1}, & \text{if } q_i = 0, \\ (a+1)^{i-s-3}, & \text{if } q_i \geq 1. \end{cases} \tag{22}$$

Proof. See Appendix B. ■

Now we use the closed-form expressions in Theorem 4 and the bounds in Lemma 5 to show that U_k can be expressed in terms of the switch size M and the parameters s, k, and m as given by (24) below in the regime that $s \geq 2$, $k \geq 2s+1$, $m \geq 2$.

Theorem 6 *Suppose that $s \geq 2$, $k \geq 2s+1$, and $m \geq 2$.*

(i) Suppose that k is even, say $k = 2\ell$ for some $\ell \geq s+1$ (note that $k \geq 2s+1$). Let q_ℓ be given as in Lemma 3. Assume that $m^{s+1} \geq q_\ell(s+1)(m-1)+1$. Then we have

$$2^{\sqrt{M \log_2(2s+2) \log_2 m/((2s+1)m)} - 4\log_2(2s+2) - (s+2)\log_2 m + 1} \leq U_k \leq 2^{\sqrt{M \log_2(2s+2) \log_2 m/((2s+1)m)} + (s+3)\log_2 m + 3}. \tag{23}$$

Therefore, in this case we have

$$U_k = 2^{O(\sqrt{M \log_2(2s+2) \log_2 m/((2s+1)m)})}. \tag{24}$$

(ii) Suppose that k is odd, say $k = 2\ell+1$ for some $\ell \geq s$ (note that $k \geq 2s+1$). Let q_ℓ be given as in Lemma 3. Assume that $m^{s+1} \geq q_{\ell+1}(s+1)(m-1)+1$. Then we have

$$2^{\sqrt{M \log_2(2s+2) \log_2 m/((2s+1)m)} - 6\log_2(2s+2) - (s+1)\log_2 m} \leq U_k \leq 2^{\sqrt{M \log_2(2s+2) \log_2 m/((2s+1)m)} + (s+3)\log_2 m + 3}. \tag{25}$$

Therefore, in this case (24) also holds.

Remark 7 *(i) As mentioned earlier in this paper, the exact result on U_k in (24) can be viewed as a complement to the approximate result in (7) in the regime that $s \geq 2$, $k \geq 2s+1$, $m \geq 2$ under the constraints in Theorem 6.*

(ii) We note that the constraints $m^{s+1} \geq q_\ell(s+1)(m-1)+1$ in Theorem 6(i) and $m^{s+1} \geq q_{\ell+1}(s+1)(m-1)+1$ in Theorem 6(ii) are not very stringent as they are applicable in practical scenarios. For example, in Table III we first find the largest q_ℓ that satisfies the constraint $m^{s+1} \geq q_\ell(s+1)(m-1)+1$, i.e., $q_\ell = \lfloor (m^{s+1}-1)/((s+1)(m-1)) \rfloor$, then we find the largest ℓ that satisfies $q_\ell(s+1)+1 \leq \ell \leq (q_\ell+1)(s+1)$, i.e., $\ell = (q_\ell+1)(s+1)$, and let $k = 2\ell = 2(q_\ell+1)(s+1)$, and finally we calculate U_k by using (14) for $s=1$ and $2 \leq m \leq 9$ and for $s=2$ and $2 \leq m \leq 5$. It is clear from Table III that for moderate values of s, m, and k (the smaller the values of s, m, and k are, the lower the construction complexity/cost of the constructions is), the maximum buffer size U_k that can be achieved is large enough and exceeds the order of millions of packets that is needed in today's commercial backbone routers. For example, for $s=1$ and $m=6$, we only need $k \geq 16$ for U_k to exceed 10^6; and for $s=2$ and $m=3$, we only need $k \geq 30$ for U_k to exceed 10^6.

(iii) Finally, we note that for the regime that $s \geq 2$, $s+1 \leq k \leq 2s$, and $m \geq 2$, we have $n = \min\{2s+1, k\}+1 = k+1$ and by using $n = k+1$ (instead of $n = 2s+2$) in the proof of Theorem 6, we can obtain the same result on U_k as that given in (6).

TABLE III

The maximum buffer size U_k given by (14) for $s=1$ and $2 \leq m \leq 9$ and for $s=2$ and $2 \leq m \leq 5$. In this table, we set $q_\ell = \lfloor (m^{s+1}-1)/((s+1)(m-1)) \rfloor$ and $k = 2(q_\ell+1)(s+1)$.

$s=1$				
m	2	3	4	5
q_ℓ	1	2	2	3
k	8	12	12	16
U_k	28	492	3270	233008

$s=1$				
m	6	7	8	9
q_ℓ	3	4	4	5
k	16	20	20	24
U_k	1.2207×10^6	1.7414×10^8	7.6896×10^8	1.7951×10^{11}

$s=2$				
m	2	3	4	5
q_ℓ	2	4	7	10
k	18	30	48	66
U_k	618	1.1488×10^7	2.0095×10^{14}	8.6281×10^{22}

Proof. **(Proof of Theorem 6)**

(i) It is clear from the definition of q_i that q_i is non-decreasing in i, and hence we see from the assumption $m^{s+1} \geq q_\ell(s+1)(m-1)+1$ that $m^{s+1} \geq q_i(s+1)(m-1)+1$ for $1 \leq i \leq \ell$. It then follows from (13), (21), (22), $1/(m-1) \leq 2/m$ (as $m \geq 2$), and $B_i \geq 1$ that

$$\max\{m^{i-s-3}, 1\} \leq B_i \leq 2m^{i-1} \text{ for } 1 \leq i \leq \ell. \tag{26}$$

It is easy to see from (14), (13), and (26) that

$$U_k \geq 2m \sum_{r=0}^{q_\ell} B_{\ell-r(s+1)} + 2q_\ell \geq 2mB_\ell \geq 2m^{\ell-s-2}, \tag{27}$$

and from (14), (13), (26), $2q_\ell+2 \leq q_\ell(s+1)+2 \leq \ell+1 \leq m^\ell$, and $m^{s+1} \geq 2^2 = 4$ that

$$\begin{aligned} U_k &\leq 2m \sum_{r=0}^{q_\ell} B_{\ell-r(s+1)} + 2q_\ell + 2 \\ &\leq 2m \sum_{r=0}^{\infty} 2m^{\ell-r(s+1)-1} + m^\ell \\ &= 4m^\ell/(1-1/m^{s+1}) + m^\ell \leq (19/3)m^\ell \leq 8m^\ell. \end{aligned} \tag{28}$$

Note that $n = \min\{2s+1, k\}+1 = 2s+2$ (as $k \geq 2s+1$). From (26), we can see that $\lceil \log_{2s+2} B_i \rceil < \log_{2s+2} B_i + 1 \leq (i-1)\log_{2s+2} m + \log_{2s+2} 2 + 1 \leq (i-1)\log_{2s+2} m + 2$ and $\lceil \log_{2s+2} B_i \rceil \geq \log_{2s+2} B_i \geq \max\{(i-s-3)\log_{2s+2} m, 0\}$ for $1 \leq i \leq \ell$. As such, we have from (2) and (13) that

$$\begin{aligned} M/m &\leq 2\sum_{i=1}^{\ell}[(2s+1)((i-1)\log_{2s+2} m + 2) + 2s + 3] \\ &= (2s+1)\ell(\ell-1)\log_{2s+2} m + (6s+5)(2\ell) \\ &\leq (2s+1)(\ell + 4/\log_{2s+2} m)^2 \log_{2s+2} m, \end{aligned} \tag{29}$$

and

$$\begin{aligned} M/m &\geq 2\sum_{i=1}^{s+2}[(2s+1)\times 0 + 2s + 3] \\ &\quad + 2\sum_{i=s+3}^{\ell}[(2s+1)(i-s-3)\log_{2s+2} m + 2s + 3] \\ &\geq (2s+1)(\ell - s - 3)^2 \log_{2s+2} m. \end{aligned} \tag{30}$$

Therefore, we see that the first inequality in (23) follows from $U_k \geq 2m^{\ell-s-2} = 2^{(\ell-s-2)\log_2 m + 1}$ in (27) and $\ell \geq \sqrt{M\log_2(2s+2)/((2s+1)m\log_2 m)} - 4\log_2(2s+2)/\log_2 m$ in (29). Similarly, we see that the second inequality in (23) follows from $U_k \leq 8m^{\ell} = 2^{\ell\log_2 m + 3}$ in (28) and $\ell \leq \sqrt{M\log_2(2s+2)/((2s+1)m\log_2 m)} + s + 3$ in (30).

(ii) As in the proof of (i) above, we can use the assumption $m^{s+1} \geq q_{\ell+1}(s+1)(m-1)+1$ to show that (26) holds for $1 \leq i \leq \ell+1$. It is then easy to see from (15), (13), and (26) that

$$\begin{aligned} U_k &\geq 2m\sum_{r=0}^{q_\ell} B_{\ell - r(s+1)} + (m-1)B_{\ell+1} + 2q_\ell + 1 \\ &\geq 2mB_\ell + (m-1)B_{\ell+1} \geq 2m^{\ell-s-2} + (m-1)m^{\ell-s-2} \\ &\geq m^{\ell-s-1}, \end{aligned} \tag{31}$$

and from (15), (13), (26), $2q_\ell + 3 \leq q_\ell(s+1) + 3 \leq \ell + 2 \leq 2m^\ell$, and $m^{s+1} \geq 2^2 = 4$ that

$$\begin{aligned} U_k &\leq 2m\sum_{r=0}^{q_\ell} B_{\ell - r(s+1)} + (m-1)B_{\ell+1} + 2q_\ell + 3 \\ &\leq 2m\sum_{r=0}^{\infty} 2m^{\ell - r(s+1) - 1} + 2(m-1)m^\ell + 2m^\ell \\ &= 4m^\ell/(1 - 1/m^{s+1}) + 2m^{\ell+1} \\ &\leq (16/3)m^\ell + 2m^{\ell+1} \leq 8m^{\ell+1}. \end{aligned} \tag{32}$$

Note that $n = \min\{2s+1, k\}+1 = 2s+2$ (as $k \geq 2s+1$). As in the proof of (i) above, we have from (2) and (13) that

$$\begin{aligned} M/m &\leq 2\sum_{i=1}^{\ell}[(2s+1)((i-1)\log_{2s+2} m + 2) + 2s + 3] \\ &\quad + (2s+1)(\ell\log_{2s+2} m + 2) + 2s + 3 \\ &= (2s+1)\ell^2\log_{2s+2} m + (6s+5)(2\ell+1) \\ &\leq (2s+1)(\ell + 6/\log_{2s+2} m)^2 \log_{2s+2} m \end{aligned} \tag{33}$$

and

$$\begin{aligned} M/m &\geq 2\sum_{i=1}^{s+2}[(2s+1)\times 0 + 2s + 3] \\ &\quad + 2\sum_{i=s+3}^{\ell}[(2s+1)(i-s-3)\log_{2s+2} m + 2s + 3] \\ &\quad + (2s+1)(\ell - s - 2)\log_{2s+2} m + 2s + 3 \\ &\geq (2s+1)(\ell - s - 2)^2 \log_{2s+2} m. \end{aligned} \tag{34}$$

Therefore, we see that the first inequality in (25) follows from $U_k \geq m^{\ell-s-1} = 2^{(\ell-s-1)\log_2 m}$ in (31) and $\ell \geq \sqrt{M\log_2(2s+2)/((2s+1)m\log_2 m)} - 6\log_2(2s+2)/\log_2 m$ in (33). Similarly, we see that the second inequality in (25) follows from $U_k \leq 8m^{\ell+1} = 2^{(\ell+1)\log_2 m + 3}$ in (32) and $\ell \leq \sqrt{M\log_2(2s+2)/((2s+1)m\log_2 m)} + s + 2$ in (34). ∎

V. Conclusion

In this paper, we have obtained a closed-form expression for the maximum buffer size that can be achieved by the constructions in [36]. Such an expression is of enough theoretical interest itself. It not only allows us to directly compute the maximum buffer size, but also makes it possible for us to give a rigorous mathematical proof that an approximate result on the maximum buffer size in [36] is indeed an exact result. Therefore, this paper complements the work in [36].

Appendix A
Proof of Lemma 3

(i) Suppose that $i \geq 3$. If $3 \leq i \leq s+1$, then we have from (10) and $2 \leq i-1 < i \leq s+1$ that

$$\begin{aligned} x_i &= \sum_{j=1}^{i-1}(ax_j + 1) = \sum_{j=1}^{i-2}(ax_j + 1) + (ax_{i-1} + 1) \\ &= x_{i-1} + (ax_{i-1} + 1) = (a+1)x_{i-1} + 1. \end{aligned}$$

On the other hand, if $i \geq s+2$, then we have from (10) and $i > i-1 \geq s+1$ that

$$\begin{aligned} x_i &= \sum_{j=i-s}^{i-1}(ax_j + 1) \\ &= \sum_{j=i-s-1}^{i-2}(ax_j + 1) + (ax_{i-1} + 1) - (ax_{i-s-1} + 1) \\ &= x_{i-1} + ax_{i-1} - ax_{i-s-1} = (a+1)x_{i-1} - ax_{i-s-1}. \end{aligned}$$

(ii) We show by induction on i that (12) holds for $i \geq 2$. It is clear from (10) that

$$x_2 = ax_1 + 1 = a \times 1 + 1 = (a^2 + a + 1)/a - 1/a.$$

Thus, we have proved the base case that (12) holds for $i = 2$ (note that $q_2 = 0$ as $0 \cdot (s+1) + 1 < 2 \leq 1 \cdot (s+1)$).

Assume as the induction hypothesis that (12) holds up to $i-1$ for some $i - 1 \geq 2$. We need to consider the following three cases.

Case 1: $q_i = 0$. In this case, we have $3 \le i \le s+1$ and hence $q_{i-1} = 0$ (as $1 < i-1 < s+1$). It follows that

$$\begin{aligned} x_i &= (a+1)x_{i-1} + 1 \\ &= (a^2+a+1)(a+1)^{i-2}/a - (a+1)/a + 1 \\ &= (a^2+a+1)(a+1)^{i-2}/a - 1/a, \end{aligned}$$

where the first equality follows from (11) (note that $3 \le i \le s+1$) and the second equality follows from the induction hypothesis that (12) holds for $i-1$ (note that $q_{i-1} = 0$). Thus, we have proved that (12) holds for i (note that $q_i = 0$).

Case 2: $q_i \ge 1$ *and* $i = q_i(s+1)+1$. In this case, we have $q_{i-1} = q_i - 1$ (as $(q_i-1)(s+1)+1 < i-1 = q_i(s+1)$) and $q_{i-s-1} = q_i - 1$ (as $(q_i-1)(s+1)+1 = i-s-1 < q_i(s+1)$).

In the following, we discuss the two subcases $q_i = 1$ and $q_i \ge 2$ separately.

Subcase 2(a): $q_i = 1$. In this subcase, we have $i = s+2$ and hence

$$\begin{aligned} x_{s+2} &= (a+1)x_{s+1} - ax_1 \\ &= (a^2+a+1)(a+1)^s/a - (a+1)/a - a \times 1 \\ &= (a^2+a+1)(a+1)^s/a - (a+1) - 1/a, \end{aligned}$$

where the first equality follows from (11) and the second equality follows from the induction hypothesis that (12) holds for $i-1 = s+1$ (note that $q_{s+1} = 0$). Thus, we have proved that (12) holds for $i = s+2$ (note that $q_{s+2} = 1$).

Subcase 2(b): $q_i \ge 2$. In this subcase, we let $y_j = i - j(s+1)$ for $0 \le j \le q_i$ and hence we have

$$\begin{aligned} x_i &= (a+1)x_{i-1} - ax_{i-s-1} \\ &= \sum_{j=0}^{q_i-1} (-1)^j (1/j!) \\ &\quad \times [j(y_j-1)_{j-1}a + (y_j-1)_j(a^2+a+1)] \\ &\quad \times a^{j-1}(a+1)^{i-j(s+1)-2} - (a+1)/a \\ &\quad - \sum_{j=0}^{q_i-1} (-1)^j (1/j!) \\ &\quad \times [j(y_{j+1})_{j-1}a + (y_{j+1})_j(a^2+a+1)] \\ &\quad \times a^j(a+1)^{i-(j+1)(s+1)-2} + 1 \\ &= (a^2+a+1)(a+1)^{i-2}/a \\ &\quad + \sum_{j=1}^{q_i-1} (-1)^j (1/j!) \\ &\quad \times \{j[(y_j-1)_{j-1} + (j-1)(y_j)_{j-2}]a \\ &\quad + [(y_j-1)_j + j(y_j)_{j-1}](a^2+a+1)\} \\ &\quad \times a^{j-1}(a+1)^{i-j(s+1)-2} \\ &\quad - (-1)^{q_i-1}a^{q_i-1}(a+1) - 1/a \\ &= \sum_{j=0}^{q_i} (-1)^j (1/j!)[j(y_j)_{j-1}a + (y_j)_j(a^2+a+1)] \\ &\quad \times a^{j-1}(a+1)^{i-j(s+1)-2} - 1/a, \end{aligned}$$

where the first equality follows from (11) (note that $i = q_i(s+1)+1 \ge 2(s+1)+1 > s+2$), the second equality follows from the induction hypothesis that (12) holds for $i-1$ and $i-s-1$ (note that $i-s-1 = (q_i-1)(s+1)+1 \ge 1\times(s+1)+1 > 2$ and $q_{i-1} = q_{i-s-1} = q_i - 1$), the third equality follows from $y_{q_i} = i - q_i(s+1) = 1$, $(1)_{q_i-2} = (q_i-2)!$, and $(1)_{q_i-1} = (q_i-1)!$, and the last equality follows from $y_{q_i} = 1$, $(1)_{q_i-1} = (q_i-1)!$, $(1)_{q_i} = (q_i)!$, and $(y-1)_{j-1} + (j-1)y_{j-2} = (y)_{j-1}$ for $1 \le j \le q_i$. Thus, we have proved that (12) holds for i.

We note that in Subcase 2(a) the induction hypothesis does not imply that (12) holds for $i = 1$ (in fact, x_1 is not given by (12)), and this is why we need to discuss the two subcases $q_i = 1$ and $q_i \ge 2$ separately.

Case 3: $q_i \ge 1$ *and* $q_i(s+1)+2 \le i \le (q_i+1)(s+1)$. In this case, we have $q_{i-1} = q_i$ (as $q_i(s+1)+1 \le i-1 < (q_i+1)(s+1)$) and $q_{i-s-1} = q_i - 1$ (as $(q_i-1)(s+1)+1 < i-s-1 \le q_i(s+1)$).

Similar to the proof in Case 2 above, we prove that (12) holds for i in this case as follows:

$$\begin{aligned} x_i &= (a+1)x_{i-1} - ax_{i-s-1} \\ &= \sum_{j=0}^{q_i} (-1)^j (1/j!) \\ &\quad \times [j(y_j-1)_{j-1}a + (y_j-1)_j(a^2+a+1)] \\ &\quad \times a^{j-1}(a+1)^{i-j(s+1)-2} - (a+1)/a \\ &\quad - \sum_{j=0}^{q_i-1} (-1)^j (1/j!) \\ &\quad \times [j(y_{j+1})_{j-1}a + (y_{j+1})_j(a^2+a+1)] \\ &\quad \times a^j(a+1)^{i-(j+1)(s+1)-2} + 1 \\ &= (a^2+a+1)(a+1)^{i-2}/a \\ &\quad + \sum_{j=1}^{q_i} (-1)^j (1/j!) \\ &\quad \times \{j[(y_j-1)_{j-1} + (j-1)(y_j)_{j-2}]a \\ &\quad + [(y_j-1)_j + j(y_j)_{j-1}](a^2+a+1)\} \\ &\quad \times a^{j-1}(a+1)^{i-j(s+1)-2} - 1/a \\ &= \sum_{j=0}^{q_i} (-1)^j (1/j!)[j(y_j)_{j-1}a + (y_j)_j(a^2+a+1)] \\ &\quad \times a^{j-1}(a+1)^{i-j(s+1)-2} - 1/a, \end{aligned}$$

where the first equality follows from (11) (note that $i \ge q_i(s+1)+2 \ge 1\times(s+1)+2 > s+2$), the second equality follows from the induction hypothesis that (12) holds for $i-1$ and $i-s-1$ (note that $i-s-1 \ge (q_i-1)(s+1)+2 \ge 2$, $q_{i-1} = q_i$, and $q_{i-s-1} = q_i - 1$), and the last equality follows from $(y-1)_{j-1} + (j-1)y_{j-2} = (y)_{j-1}$ for $1 \le j \le q_i + 1$.

(iii) To show that the sequence $\{x_i\}_{i=1}^{\infty}$ is strictly increasing, we prove by induction on i that $x_1 < x_2 < \cdots < x_i$ for $i \ge 2$. It is clear from (10) and $a > 0$ that

$$x_2 - x_1 = (ax_1 + 1) - x_1 = a \times 1 + 1 - 1 = a > 0.$$

Thus, we have proved the base case that $x_1 < x_2$.

Assume as the induction hypothesis that $x_1 < x_2 < \cdots < x_{i-1}$ for some $i-1 \ge 2$. To complete the induction, it suffices to prove that $x_{i-1} < x_i$. We consider the following two cases.

Case 1: $3 \le i \le s+1$. In this case, we have

$$x_i - x_{i-1} = ax_{i-1} + 1 > ax_1 + 1 = a \times 1 + 1 > 0,$$

where the equality follows from (11), the first inequality follows from $a > 0$ and the induction hypothesis that $x_{i-1} > x_1$ (note that $i-1 > 1$), and the last inequality follow from $a > 0$.

Case 2: $i \geq s+2$. In this case, we have

$$x_i - x_{i-1} = a(x_{i-1} - x_{i-s-1}) > 0,$$

where the equality follows from (11), and the inequality follows from $a > 0$ and the induction hypothesis that $x_{i-1} > x_{i-s-1}$ (note that $1 \leq i-s-1 < i-1$).

APPENDIX B
PROOF OF LEMMA 5

Suppose that $(a+1)^{s+1} \geq q_i(s+1)a + 1$ for some $i \geq 1$. If $i = 1$, then $q_i = 0$ and it is clear from $x_1 = 1$ and $a > 0$ that (21) and (22) hold for $i = 1$. Therefore, we assume that $i \geq 2$ in the rest of the proof.

Write x_i in (12) as follows:

$$x_i = \sum_{j=0}^{q_i} (-1)^j \delta_j - 1/a, \tag{35}$$

where

$$\begin{aligned}\delta_j = (1/j!)[&j(i-j(s+1))_{j-1}a \\ &+(i-j(s+1))_j(a^2+a+1)] \\ &\times a^{j-1}(a+1)^{i-j(s+1)-2},\end{aligned} \tag{36}$$

for $0 \leq j \leq q_i$. Note that it is easy to see that $\delta_j > 0$ (as $i - j(s+1) \geq i - q_i(s+1) \geq 1$ and $a > 0$) for $0 \leq j \leq q_i$. Then we consider the cases $q_i = 0$ and $q_i \geq 1$ separately.

Case 1: $q_i = 0$. In this case, we have from (35), (36), $a > 0$, and $i \geq 2$ that

$$\begin{aligned}x_i &= \delta_0 - 1/a = (a^2+a+1)(a+1)^{i-2}/a - 1/a \\ &= \begin{cases}(a+1)^i/a - (a+1)^{i-2} - 1/a \leq (a+1)^i/a, \\ (a+1)^{i-1} + (a+1)^{i-2}/a - 1/a \geq (a+1)^{i-1}.\end{cases}\end{aligned} \tag{37}$$

Thus, (21) and (22) hold in this case.

Case 2: $q_i \geq 1$. In this case, we first show that the sequence $\{\delta_j\}_{j=1}^{q_i}$ is strictly decreasing. Suppose $1 \leq j \leq q_i - 1$. Note that

$$\begin{aligned}\delta_{j+1} = (1/(j+1)!)[&(j+1)(i-(j+1)(s+1))_j a \\ &+(i-(j+1)(s+1))_{j+1}(a^2+a+1)] \\ &\times a^j(a+1)^{i-(j+1)(s+1)-2}.\end{aligned} \tag{38}$$

To show that $\delta_j > \delta_{j+1}$, we give upper bounds for the two terms $(i-(j+1)(s+1))_j$ and $(i-(j+1)(s+1))_{j+1}$ that appear in the expression for δ_{j+1} in (38). Specifically, we have

$$\begin{aligned}&(i-(j+1)(s+1))_j \\ &= (i-(j+1)(s+1))_{j-1}(i-(j+1)(s+1)+j-1) \\ &\leq (i-j(s+1))_{j-1}(q_i-1)(s+1),\end{aligned} \tag{39}$$

where the inequality follows from $(i-j(s+1))_{j-1} \geq (i-(j+1)(s+1))_{j-1} > 0$ (as $1 \leq j \leq q_i - 1$ and $i - j(s+1) > i-(j+1)(s+1) \geq i - q_i(s+1) \geq 1$) and $0 < i-(j+1)(s+1)+j-1 \leq (q_i+1)(s+1)-(j+1)(s+1)+j-1 = (q_i-1)(s+1)-(j-1)s \leq (q_i-1)(s+1)$. Similarly we have

$$\begin{aligned}&(i-(j+1)(s+1))_{j+1} \\ &= (i-(j+1)(s+1))_j(i-(j+1)(s+1)+j) \\ &\leq (i-j(s+1))_j((q_i-1)(s+1)+1).\end{aligned} \tag{40}$$

As such, we have

$$\begin{aligned}&\delta_j - \delta_{j+1} \\ &= (1/j!)[j(i-j(s+1))_{j-1}a \\ &\quad +(i-j(s+1))_j(a^2+a+1)]a^{j-1}(a+1)^{i-j(s+1)-2} \\ &\quad -(1/(j+1)!)[(j+1)(i-(j+1)(s+1))_j a \\ &\quad +(i-(j+1)(s+1))_{j+1}(a^2+a+1)] \\ &\quad \times a^j(a+1)^{i-(j+1)(s+1)-2} \\ &\geq (1/j!)[j(a+1)^{s+1} - (q_i-1)(s+1)a] \\ &\quad \times (i-j(s+1))_{j-1}a^j(a+1)^{i-(j+1)(s+1)-2} \\ &\quad +(1/(j+1)!) \\ &\quad \times[(j+1)(a+1)^{s+1} - ((q_i-1)(s+1)+1)a] \\ &\quad \times (i-j(s+1))_j(a^2+a+1)a^{j-1}(a+1)^{i-(j+1)(s+1)-2} \\ &> 0,\end{aligned} \tag{41}$$

where the equality follows from (36) and (38), the first inequality follows from $a > 0$, (39), and (40), and the last inequality follows from $j \geq 1$ and $(a+1)^{s+1} \geq q_i(s+1)a+1$.

Now we use the strict monotonicity of the sequence $\{\delta_j\}_{j=1}^{q_i}$ in (41) to show that

$$\delta_0 - \delta_1 - 1/a \leq x_i \leq \delta_0 - 1/a. \tag{42}$$

Specifically, if q_i is odd, say $q_i = 2\ell - 1$ for some $\ell \geq 1$, then it follows from (35), the strict monotonicity of the sequence $\{\delta_j\}_{j=1}^{q_i}$, and the positivity of the $\delta_j's$ that

$$x_i = \delta_0 - \sum_{j=1}^{\ell-1}(\delta_{2j-1} - \delta_{2j}) - \delta_{2\ell-1} - 1/a \leq \delta_0 - 1/a,$$

and

$$x_i = \delta_0 - \delta_1 + \sum_{j=1}^{\ell-1}(\delta_{2j} - \delta_{2j+1}) - 1/a \geq \delta_0 - \delta_1 - 1/a.$$

Similarly, if q_i is even, say $q_i = 2\ell$ for some $\ell \geq 1$, then we also have

$$x_i = \delta_0 - \sum_{j=1}^{\ell}(\delta_{2j-1} - \delta_{2j}) - 1/a \leq \delta_0 - 1/a,$$

and

$$\begin{aligned}x_i &= \delta_0 - \delta_1 + \sum_{j=1}^{\ell-1}(\delta_{2j} - \delta_{2j+1}) + \delta_{2\ell} - 1/a \\ &\geq \delta_0 - \delta_1 - 1/a.\end{aligned}$$

Finally, we have from (42) and $\delta_0 - 1/a \leq (a+1)^i/a$ in (37) that

$$x_i \leq \delta_0 - 1/a \leq (a+1)^i/a.$$

Thus, (21) holds in this case. To show that (22) holds in this case, we note from (42) and (36) that

$$\begin{aligned}
&x_i \\
&\geq \delta_0 - \delta_1 - 1/a \\
&= (a^2+a+1)(a+1)^{i-2}/a \\
&\quad -[a+(i-s-1)(a^2+a+1)](a+1)^{i-s-3} - 1/a \\
&= [(a+1)^{s+1} - (i-s-1)a](a^2+a+1)(a+1)^{i-s-3}/a \\
&\quad -a(a+1)^{i-s-3} - 1/a. \qquad (43)
\end{aligned}$$

If $q_i(s+1)+1 \leq i \leq (q_i+1)(s+1)-1$, then we have $(a+1)^{s+1} - (i-s-1)a \geq q_i(s+1)a + 1 - (q_i(s+1)-1)a = a+1$, and hence it follows from (43), $a > 0$, and $i-s-2 \geq (q_i-1)(s+1) \geq 0$ that

$$\begin{aligned}
x_i &\geq (a^2+a+1)(a+1)^{i-s-2}/a - a(a+1)^{i-s-3} - 1/a \\
&= (a+1)^{i-s-3} + (a^2+1)(a+1)^{i-s-2}/a - 1/a \\
&\geq (a+1)^{i-s-3}. \qquad (44)
\end{aligned}$$

On the other hand, if $i = (q_i+1)(s+1)$, then we have $(a+1)^{s+1} - (i-s-1)a \geq q_i(s+1)a + 1 - q_i(s+1)a = 1$, and hence it follows from (43), $a > 0$, and $i-s-3 = q_i(s+1) - 2 \geq 1 \times 2 - 2 = 0$ that

$$\begin{aligned}
x_i &\geq (a^2+a+1)(a+1)^{i-s-3}/a - a(a+1)^{i-s-3} - 1/a \\
&= (a+1)^{i-s-3} + (a+1)^{i-s-3}/a - 1/a \\
&\geq (a+1)^{i-s-3}. \qquad (45)
\end{aligned}$$

Thus, we see from (44) and (45) that (22) holds in this case.

REFERENCES

[1] M. J. Karol, "Shared-memory optical packet (ATM) switch," in *Proc. SPIE: Multigigabit Fiber Communication Systems*, 1993.

[2] Z. Hass, "The "staggering switch": An electronically controlled optical packet switch," *J. Lightw. Technol.*, vol. 11, pp. 925–936, May/Jun. 1993.

[3] I. Chlamtac and A. Fumagalli, "Quadro-star: A high performance optical WDM star network," *IEEE Trans. Commun.*, vol. 42, pp. 2582–2591, Aug. 1994.

[4] I. Chlamtac *et al.*, "Cord: contention resolution by delay lines," *IEEE J. Sel. Areas Commun.*, vol. 14, pp. 1014–1029, Jun. 1996.

[5] R. L. Cruz and J.-T. Tsai, "COD: alternative architectures for high speed packet switching," *IEEE/ACM Trans. Netw.*, vol. 4, pp. 11–21, Feb. 1996.

[6] D. K. Hunter *et al.*, "2×2 buffered switch fabrics for traffic routing, merging and shaping in photonic cell networks," *J. Lightw. Technol.*, vol. 15, pp. 86–101, Jan. 1997.

[7] D. K. Hunter, W. D. Cornwell, T. H. Gilfedder, A. Franzen, and I. Andonovic, "SLOB: A switch with large optical buffers for packet switching," *J. Lightw. Technol.*, vol. 16, pp. 1725–1736, Oct. 1998.

[8] E. Varvarigos, "The "packing" and the "scheduling packet" switch architectures for almost all-optical lossless networks," *J. Lightw. Technol.*, vol. 16, pp. 1757–1767, Oct. 1998.

[9] I. Chlamtac, A. Fumagalli, and C.-J. Suh, "Multibuffer delay line architectures for efficient contention resolution in optical switching nodes," *IEEE Trans. Commun.*, vol. 48, pp. 2089–2098, Dec. 2000.

[10] Y.-T. Chen, C.-S. Chang, J. Cheng, and D.-S. Lee, "Feedforward SDL constructions of output-buffered multiplexers and switches with variable length bursts," in *Proc. IEEE INFOCOM*, 2007.

[11] I. Chlamtac, A. Fumagalli, and C.-J. Suh, "Optimal 2×1 multi-stage optical packet multiplexer," in *Proc. IEEE GLOBECOM*, 1997.

[12] C.-S. Chang, D.-S. Lee, and C.-K. Tu, "Recursive construction of FIFO optical multiplexers with switched delay lines," *IEEE Trans. Inf. Theory*, vol. 50, pp. 3221–3233, Dec. 2004.

[13] C.-S. Chang, D.-S. Lee, and C.-K. Tu, "Using switched delay lines for exact emulation of FIFO multiplexers with variable length bursts," *IEEE J. Selected Areas Commun.*, vol. 24, pp. 108–117, Apr. 2006.

[14] C.-C. Chou, C.-S. Chang, D.-S. Lee, and J. Cheng, "A necessary and sufficient condition for the construction of 2-to-1 optical FIFO multiplexers by a single crossbar switch and fiber delay lines," *IEEE Trans. Inf. Theory*, vol. 52, pp. 4519–4531, Oct. 2006.

[15] J. Cheng, "Constructions of fault tolerant optical 2-to-1 FIFO multiplexers," *IEEE Trans. Inf. Theory*, vol. 53, pp. 4092–4105, Nov. 2007.

[16] J. Cheng, "Constructions of optical 2-to-1 FIFO multiplexers with a limited number of recirculations," *IEEE Trans. Inf. Theory*, vol. 54, pp. 4040–4052, Sep. 2008.

[17] J. Cheng *et al.*, "Greedy constructions of optical queues with a limited number of recirculations," *IEEE Trans. Inf. Theory*, vol. 63, pp. 5314–5326, Aug. 2017.

[18] X. Wang, X. Jiang, and S. Horiguchi, "Improved bounds on the feedfoward design of optical multiplexers," in *Proc. I-SPAN*, 2008.

[19] X. Wang, X. Jiang, and A. Pattavina, "Constructing N-to-N shared optical queues with switches and fiber delay lines," *IEEE Trans. Inf. Theory*, vol. 58, pp. 3836–3842, Jun. 2012.

[20] S.-Y. R. Li and X. J. Tan, "Mux/demux queues, FIFO queues, and their construction by fiber memories," *IEEE Trans. Inf. Theory*, vol. 57, pp. 1328–1343, Mar. 2011.

[21] C.-S. Chang, Y.-T. Chen, and D.-S. Lee, "Constructions of optical FIFO queues," *IEEE Trans. Inf. Theory*, vol. 52, pp. 2838–2843, Jun. 2006.

[22] B. A. Small, A. Shacham, and K. Bergman, "A modular, scalable, extensible, and transparent optical packet buffer," *J. Lightw. Technol.*, vol. 25, pp. 978–985, Apr. 2007.

[23] P.-K. Huang, C.-S. Chang, J. Cheng, and D.-S. Lee, "Recursive constructions of parallel FIFO and LIFO queues with switched delay lines," *IEEE Trans. Inf. Theory*, vol. 53, pp. 1778–1798, May 2007.

[24] N. Beheshti and Y. Ganjali, "Packet scheduling in optical FIFO buffers," in *Proc. IEEE HSN*, 2007.

[25] X. Wang, X. Jiang, A. Pattavina, and S. Horiguchi, "A construction of 1-to-2 shared optical buffer queue with switched delay lines," *IEEE Trans. Commun.*, vol. 57, pp. 3712–3723, Dec. 2009.

[26] X. Wang, X. Jiang, and A. Pattavina, "Efficient designs of optical LIFO buffer with switches and fiber delay lines," *IEEE Trans. Commun.*, vol. 59, pp. 3430–3439, Dec. 2011.

[27] A. D. Sarwate and V. Anantharam, "Exact emulation of a priority queue with a switch and delay lines," *Queueing Systems: Theory Appl.*, vol. 53, pp. 115–125, Jul. 2006.

[28] H.-C. Chiu, C.-S. Chang, J. Cheng, and D.-S. Lee, "A simple proof for the constructions of optical priority queues," *Queueing Systems: Theory Appl.*, vol. 56, pp. 73–77, Jun. 2007.

[29] H.-C. Chiu, C.-S. Chang, J. Cheng, and D.-S. Lee, "Using a single switch with $O(M)$ inputs/outputs for the construction of an optical priority queue with $O(M^3)$ buffer," in *Proc. IEEE INFOCOM*, 2007.

[30] H. Kogan and I. Keslassy, "Optimal-complexity optical router," in *Proc. IEEE INFOCOM*, 2007.

[31] H. Rastegarfar, M. Ghobadi, and Y. Ganjali, "Emulation of Optical PIFO Buffers," in *Proc IEEE GLOBECOM*, 2009.

[32] J. Cheng, H.-C. Chiu, C.-S. Chang, and D.-S. Lee, "Constructions of optical priority queues with multiple inputs and multiple outputs," *IEEE Trans. Inf. Theory*, vol. 57, pp. 4274–4301, Jul. 2011.

[33] J. Cheng *et al.*, "Average number of recirculations in SDL constructions of optical priority queues," *IEEE Commun. Lett.*, vol. 15, pp. 899–901, Aug. 2011.

[34] A. Datta, "Construction of polynomial-size optical priority queues using linear switches and fiber delay lines," *IEEE/ACM Trans. Netw.*, vol. 25, pp. 974–987, Apr. 2017.

[35] B. Tang, X. Wang, C.-T. Nguyen, B. Ye, and S. Lu, "Construction of subexponential-size optical priority queues with switches and fiber delay lines," *IEEE/ACM Trans. Netw.*, vol. 28, pp. 336–346, Feb. 2020.

[36] J. Cheng, S.-H. Yang, C.-Y. Wang, H.-H. Tang, and B. Tang, "On efficient constructions of optical priority queues," *IEEE Trans. Commun.*, vol. 70, pp. 1861–1874, Mar. 2022.

[37] F. Jordan, D. Lee, K. Y. Lee, and S. V. Ramanan, "Serial array time slot interchangers and optical implementations," *IEEE Trans. Computers*, vol. 43, pp. 1309–1318, 1994.

[38] Y.-T. Chen, J. Cheng, and D.-S. Lee, "Constructions of linear compressors, non-overtaking delay lines, and flexible delay lines for optical packet switching," *IEEE/ACM Trans. Netw.*, vol. 17, pp. 2014–2027, Dec. 2009.

[39] C.-S. Chang, J. Cheng, T.-H. Chao, and D.-S. Lee, "Optimal constructions of fault tolerant optical linear compressors and linear decompressors," *IEEE Trans. Commun.*, vol. 57, pp. 1140–1150, Apr. 2009.

[40] T.-H. Chao, C.-S. Chang, D.-S. Lee, and J. Cheng, "Constructions of multicast flexible delay lines and optical multicast switches with 100% throughput," in *Proc. IEEE GLOBECOM*, 2007.

[41] H.-W. Lan, C.-S. Chang, J. Cheng, and D.-S. Lee, "Constructions and analysis of crosstalk-free optical queues," in *Proc. IEEE HPSR*, 2008.
[42] D.-S. Lee, K.-J. Hsu, C.-S. Chang, and J. Cheng, "Emulation and approximation of a flexible delay line by parallel non-overtaking delay lines," in *Proc. IEEE INFOCOM*, 2009.
[43] D.-S. Lee, C.-S. Chang, J. Cheng, H.-S. Chueh, and K.-T. Wang, "Emulation of an optical flexible delay line by parallel variable optical delay lines," *IEEE Commun. Lett.*, vol. 14, pp. 770–772, Aug. 2010.
[44] C.-S. Chang, J. Cheng, and D.-S. Lee, "SDL constructions of FIFO, LIFO and absolute contractors," in *Proc. IEEE INFOCOM*, 2009.
[45] H. Kogan and I. Keslassy, "Fundamental complexity of optical systems," in *Proc. IEEE INFOCOM*, 2007.
[46] D.-S. Lee, C.-S. Chang, J. Cheng, and H.-S. Yan, "Queueing analysis of loss systems with variable optical delay lines," in *Proc. IEEE INFOCOM*, 2008.
[47] J. Liu, T. T. Lee, X. Jiang, and S. Horiguchi, "Blocking and delay analysis of single wavelength optical buffer with general packet size distribution," *J. Lightw. Technol.*, vol. 27, pp. 955–966, Apr. 2009.
[48] D. K. Hunter, M. C. Chia, and I. Andonovic, "Buffering in optical packet switches," *J. Lightw. Technol.*, vol. 16, pp. 2081–2094, Dec. 1998.
[49] S. Yao, B. Mukherjee, and S. Dixit, "Advances in photonic packet switching: An overview," *IEEE Commun. Mag.*, vol. 38, pp. 84–94, Feb. 2000.
[50] D. K. Hunter and I. Andonovic, "Approaches to optical Internet packet switching," *IEEE Commun. Mag.*, vol. 38, pp. 116–122, Sep. 2000.
[51] S. Yao, B. Mukherjee, S. J. B. Yoo, and S. Dixit, "A unified study of contention-resolution schemes in optical packet-switched networks," *J. Lightw. Technol.*, vol. 21, pp. 672–683, Mar. 2003.
[52] S. J. B. Yoo, "Optical packet and burst switching technologies for the future photonic Internet," *J. Lightw. Technol.*, vol. 24, pp. 4468–4492, Dec. 2006.
[53] K. Miklós, "Congestion resolution and buffering in packet switched all-optical networks," Ph.D. Dissertation, Budapest University of Techonology and Economics, Budapest, Hungary, 2008.

Jay Cheng (S'00-M'03-SM'09) received the B.S. and M.S. degrees from National Tsing Hua University, Hsinchu, Taiwan, in 1993 and 1995, respectively, and the Ph.D. degree from Cornell University, Ithaca, NY, USA, in 2003, all in Electrical Engineering. Since August 2003, he has been with the Department of Electrical Engineering at National Tsing Hua University, Hsinchu, Taiwan, where he is currently a Professor. His current research interests include optical queueing theory, high-speed switching, network science, wireless communications, and information theory.

Shin-Shiang Huang received the B.S. degree in Industrial Engineering and Management from National Chiao Tung University, Hsinchu, Taiwan, in 2019 and the M.S. degree in Industrial Engineering and Engineering Management from National Tsing Hua University, Hsinchu, Taiwan, in 2021. He is currently a Ph.D. student at the Department of Electrical Engineering, National Tsing Hua University, Hsinchu, Taiwan. His current research interest is in optical queueing theory.

Hsin-Hung Chou received the B.S. degree in Electrical Engineering and the M.S. degree in Communications Engineering from National Tsing Hua University, Hsinchu, Taiwan, in 2004 and 2006, respectively. From 2014 to 2017, he was with MeidaTek Inc., Hsinchu, Taiwan, developing 4G LTE baseband receiver algorithm design. Since 2018, he has been with HON LIN Technology Co., Ltd, Hsinchu, Taiwan, focusing on 5G O-RAN small cell design. His current research interests include optical queueing theory and PHY-MAC cross-layer design for wireless networks.

Ming-Che Tang received the B.S. degree in Electrical Engineering from National Tsing Hua University, Hsinchu, Taiwan, in 2022. He is currently an M.S. student at the Department of Electrical Engineering, National Tsing Hua University, Hsinchu, Taiwan. His current research interest is in optical queueing theory.